Statistics of Mass Aggregation in a Self-Gravitating One-Dimensional Gas

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We study at the microscopic level the dynamics of a one-dimensional, gravitationally interacting sticky gas. Initially, N identical particles of mass m with uncorrelated, randomly distributed velocities fill homogeneously a finite region of space. It is proved that at a characteristic time a single macroscopic mass is formed with certainty, surrounded by a dust of nonextensive fragments. In the continuum limit this corresponds to a single shock creating a singular mass density. The statistics of the remaining fragments obeys the Poisson law at all times following the shock. Numerical simulations indicate that up to the moment of macroscopic aggregation the system remains internally homogeneous. At the short time scale a rapid decrease in the kinetic energy is observed, accompanied by the formation of a number $\sim \sqrt{N}$ of aggregates with masses $\sim m\sqrt{N}$.

1. INTRODUCTION

The model of gravitationally interacting sticky particles has been proposed by Zeldovich,[1] and then extensively studied in connection with the problem of large scale structures in the universe (for a recent review see ref. 2). The effort has been concentrated on the study of solutions of the corresponding self-consistent hydrodynamic equations in order to understand how small density fluctuations arround a homogeneous state could induce observed mass distributions.

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In this paper we analyze the dynamics of a one-dimensional system with an initial homogeneous mass distribution filling only a finite region of space. As the one-dimensional gravitational interaction is confining, the whole system will eventually form a single mass. We start from the microscopic dynamics of N point particles with randomly chosen initial velocities. Then we determine the evolution of the statistical distribution of masses formed by merging at binary sticky collisions. It turns out that a macroscopic mass is formed after a characteristic finite time with probability one, surrounded by a cloud of non-extensive fragments. In the continuum limit, this corresponds to a solution of the system of mass and momentum conservation laws, showing a single shock at the characteristic time. We emphasize that energy is not conserved in this process. We thus find not only a simple realization of a particular solution belonging to the general class studied in ref. 3, but also a detailed description of the statistics of masses in the course of time.

As mentioned above, the system is composed of point particles attracting each other with forces proportional to the product of their masses, and independent of the interparticle distance. The corresponding n-body Hamiltonian has the form

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + \gamma \sum_{i<j}^{n} \sum^{n} m_i m_j \, |x_i - x_j| \tag{1}$$

where γ is the gravitational constant, and m_i, x_i, p_i denote the mass, the position and the momentum of particle i, respectively.

The Hamiltonian (1) with an appropriate value of n defines the dynamics of the system in the time intervals separating binary collisions. The collisions, which are supposed to be perfectly inelastic, are responsible for the mass aggregation. When two neighbouring particles i and j meet, they merge instantaneously forming a single mass $(m_i + m_j)$ which continues the motion acquiring at the moment of impact the momentum $(p_i + p_j)$. The complete dynamics, involving the aggregation process, is thus subject to the mass and momentum conservation laws. On the other hand, the number of particles decreases monotonically in the course of time, each collision replacing two particles by one.

In a previous work[4] we concentrated on the determination of the probability $P_N(t)$ of merging before time t of the initial N-particle dust into a single body (we denote here by N the number of particles at time $t = 0$). The rigorous results have been obtained by assuming a uniform equidistant configuration of the initial identical masses m, characterized by a constant mass density

$$\rho = \frac{m}{a} \tag{2}$$

where a is the distance between the nearest neighbours. The velocities of the particles were supposed to be uncorrelated at $t = 0$, distributed according to some probability density $\phi(v)$. In particular, the Gaussian law

$$\phi_\lambda(v) = \frac{1}{\sqrt{2\pi}\,\lambda} \exp\left(-\frac{v^2}{2\lambda^2}\right) \tag{3}$$

has been considered in a detailed way.

It has been found that for a macroscopic amount of matter $N \to \infty$ the probability $P_N(t)$ of such a complete merging vanished for times shorter that the characteristic time

$$t^* = \frac{1}{\sqrt{\gamma\rho}} \tag{4}$$

More precisely, expressing the probability $P_N(t)$ in terms of the relevant time variable

$$\tau = \gamma t - \frac{1}{\rho t} \tag{5}$$

we derived a general (independent of the form of distribution ϕ), remarkably simple result

$$P_N(\tau = 0) = \frac{1}{N} \tag{6}$$

from which the above assertion followed.

The exact form of the monotonically increasing function $P_N(\tau)$ was then determined in the limit of a continuous initial mass distribution. One way of defining this limit is given below:

$$N \to \infty, \qquad m \to 0, \qquad a \to 0$$

$$M_{\text{tot}} = Nm = \text{const}, \qquad \rho = \frac{m}{a} = \text{const}, \qquad \lambda = \text{const} \tag{7}$$

However, it has been stressed in our concluding remarks that the really relevant question was that of the macroscopic aggregation, which is quite different from the complete merging into a single mass. Indeed, from the physical point of view the important event is certainly that of the

formation of a macroscopic mass, representing a finite fraction of the total mass $M_{tot} = Nm$ in the continuum limit (7). And this event, according to numerical simulations and preliminary analytical results, seemed to occur in the immediate vicinity of the characteristic time t^*, which could not be predicted from the knowledge of the probability $P_N(\tau)$. The main object of the present study is to clarify this point by analyzing the evolution of the mass distribution in the course of time. In other words, we determine here the statistics of the mass aggregation providing a proof that the mass density in the continuum limit (7) becomes singular after a finite time t^*, and takes the form of the Dirac δ centered on a position of a single macroscopic mass.

The concentration of the mass density on a single point at time t^* can be simply demonstrated in the special case of a static initial condition, when all the particles at $t = 0$ are at rest: $\phi(v) = \delta(v)$. Indeed, suppose that the initial positions of the particles are $x_j(0) = ja$, $j = 1, 2,..., N$. Then, at time $t > 0$, the distances between neighbouring particles shrink to $(a - m\gamma t^2)$, so that all the particles merge simultaneously at the moment $t = t^* = \sqrt{a/\gamma m}$. The initial mass density uniformly distributed within the interval $0 < x < Na$ is given by

$$\rho(x; 0) = \theta(x)\,\theta(Na - x)\,\frac{m}{a} = \theta(x)\,\theta\left(\frac{M_{tot}}{\rho} - x\right)\rho$$

where $\theta(x)$ is the Heaviside unit step function. For times $0 < t < t^*$, it acquires in the continuum limit (7) the value

$$\rho(x; t) = \theta\left[x - \gamma M_{tot}\frac{t^2}{2}\right]\theta\left[\frac{M_{tot}}{\rho} - \gamma M_{tot}\frac{t^2}{2} - x\right]\frac{\rho}{1 - \rho\gamma t^2} \qquad (8)$$

Equation (8) implies the expected result

$$\lim_{t \to t^*} \rho(x; t) = M_{tot}\,\delta\left(x - \frac{M_{tot}}{2\rho}\right) \qquad (9)$$

It is shown in the present paper how the above result can be generalized to the case of a random (Gaussian) initial velocity distribution (3), reflecting the creation of a macroscopic body at $t = t^*$. Moreover, we derive an analytic formula for the distribution of the remaining nonextensive microscopic masses. Computer simulations supplement our analytic approach on two points. They permit to determine the scaling laws in the approach to the continuum limit and to predict the mass distribution before t^*, a problem which remains open for the mathematical analysis.

Before closing these introductory remarks let us recall that the system of mass and momentum conservation laws governing the dynamics of one-dimensional aggregation has been recently studied from the point of view of the existence of global weak solutions.[3] In their proof the authors have analyzed the dynamics in terms of the center of mass trajectories in much the same way as it had already been done in our previously published papers.[4, 5]

2. EVALUATING THE MASS DENSITY

At the initial moment $t = 0$, N identical masses m start the motion with uncorrelated velocities, distributed according to the probability density $\phi(v)$. Particle j begins to move from the point ja, $j = 1, 2,..., N$. In order to study the evolution of the mass density one has to determine the probability density for finding at time $t > 0$ a mass M at point X. The dynamics of the system implies that M results from aggregation of some cluster of neighbouring initial masses, and that X must coincide with the position of the center of mass of the cluster at time t. So, let us consider a n-particle cluster

$$(j+1, j+2,..., j+n) \tag{10}$$

In accordance with the dynamics induced by the Hamiltonian (1) its center of mass X_{j+1}^n follows the trajectory

$$X_{j+1}^n(t) = \left[j + \frac{(n+1)}{2} \right] a + \frac{t}{n} \sum_{s=1}^{n} v_{j+s} + \gamma m(N-n-2j)\frac{t^2}{2} \tag{11}$$

with velocity

$$V_{j+1}^n(t) = \frac{1}{n} \sum_{s=1}^{n} v_{j+s} + \gamma m(N-n-2j) t \tag{12}$$

The n particles (10) merge into a single mass nm before time t if and only if

$$X_{j+1}^r(t) > X_{j+r+1}^{n-r}(t), \qquad r = 1, 2,..., n-1 \tag{13}$$

The inequalities (13) express the requirement that for any partition of the n-particle cluster (10) into subclusters $(j+1, j+2,..., j+r)$ and $(j+r+1, j+r+2,..., n)$ the centers of mass of the subclusters cross before time t

leading to the total merging. In order to guarantee that a single mass $M_n = nm$ is actually observed at the point (11) we have still to rule out the possibility of disturbance which would cause collisions with surrounding masses, built up from the initial clusters

$$(1, 2, ..., j) \tag{14}$$

and

$$(n + 1, n + 2, ..., N) \tag{15}$$

The unperturbed aggregation of the n-particle cluster (10) occurs if and only if

$$X^s_{j-s+1}(t) < X^n_{j+1}(t), \qquad s = 1, 2, ..., j \tag{16}$$

$$X^n_{j+1}(t) < X^s_{n+1}(t), \qquad s = 1, 2, ..., N - j - n \tag{17}$$

We continue to use here the notation X^n_{i+1} to denote the position of the center of mass of the n-particle cluster composed of particles $(i + 1, i + 2, ..., i + n)$.

The inequalities (16) express the fact that the center of mass trajectories of the s-particle clusters

$$(j - s + 1, j - s + 2, ..., j), \qquad s = 1, 2, ..., j$$

stay to the left of the trajectory $X^n_{j+1}(t)$ up to time t, and thus do not cross it. Similarly, the inequalities (17) exclude crossing of the trajectory of the aggregating mass M_n with the center of mass trajectories on which evolve masses (15) initially to the right of it. The necessary and sufficient character of the conditions (16) and (17) follows from the remark that the dynamics excludes more than one crossing between the particle trajectories.

The probability density for finding at time $t > 0$ a mass M at point X with velocity V can now be written in the form

$$\left\langle \sum_{n=1}^{N} \sum_{j=0}^{N-n} \delta[X - X^n_{j+1}(t)] \, \delta[V - V^n_{j+1}(t)] \, \delta(M - nm) \right.$$

$$\times \prod_{r=1}^{n-1} \theta[X^r_{j+1}(t) - X^{n-r}_{j+r+1}(t)]$$

$$\left. \times \prod_{s=1}^{j} \theta[X^n_{j+1}(t) - X^s_{j-s+1}(t)] \prod_{s=1}^{N-j-n} \theta[X^s_{n+1}(t) - X^n_{j+1}(t)] \right\rangle \tag{18}$$

where $\langle \cdots \rangle$ denotes the mean value with respect to the initial velocity distribution

$$\prod_{i=1}^{N} \phi(v_i) \tag{19}$$

Multiplying formula (18) by M, and integrating over all possible masses and velocities we arrive at the expression for the mass density

$$\rho(X; t) = \sum_{n=1}^{N} \sum_{j=0}^{N-n} nm \left\langle \delta[X - X_{j+1}^{n}(t)] \prod_{r=1}^{n-1} \theta[X_{j+1}^{r}(t) - X_{j+r+1}^{n-r}(t)] \right\rangle$$

$$\times \left\langle \prod_{s=1}^{j} \theta[X - X_{j-s+1}^{s}(t)] \right\rangle \left\langle \prod_{s=1}^{N-j-n} \theta[X_{n+1}^{s}(t) - X] \right\rangle \tag{20}$$

In writing Eq. (20) we took into account the absence of correlations in the velocity distribution (19). As a result the mean value appearing in Eq. (18) factorized out into the product of three averages corresponding to disjoint groups of velocity variables $(v_1,..., v_j)$, $(v_{j+1},..., v_{j+n})$ and $(v_{j+n+1},..., v_N)$. The calculations greatly simplify in the case of the Gaussian form (3) which will be used in the sequel. In order to show the kind of problems one has to deal with let us consider in Eq. (20) the integration over the velocities $(v_{j+1}, v_{j+2},..., v_{j+n})$ of the aggregating n-particle cluster. Using Eq. (11) we find

$$\left\langle \delta[X - X_{j+1}^{n}(t)] \prod_{r=1}^{n-1} \theta[X_{j+1}^{r}(t) - X_{j+r+1}^{n-r}(t)] \right\rangle$$

$$= \int dv_{j+1} \cdots \int dv_{j+n} \, \phi_\lambda(v_{j+1}) \cdots \phi_\lambda(v_{j+n})$$

$$\times \delta\left[X - \left(j + \frac{(n+1)}{2}\right)a - \frac{t}{n} \sum_{s=1}^{n} v_{j+s} - \gamma m(N - n - 2j)\frac{t^2}{2} \right]$$

$$\times \prod_{r=1}^{n-1} \theta\left\{ \sum_{s=1}^{r} v_{j+s} - \frac{r}{n} \sum_{s=1}^{n} v_{j+s} + \frac{r(n-r)}{2} m\tau \right\} \tag{21}$$

By introducing new integration variables

$$u_r = \frac{1}{\lambda}\left(\sum_{s=1}^{r} v_{j+s} - \frac{r}{n} \sum_{s=1}^{n} v_{j+s} \right), \qquad r = 1, 2,..., n-1$$

$$u_n = \frac{1}{\lambda} \sum_{s=1}^{n} v_{j+s} \tag{22}$$

one eventually rewrites formula (21) in the form

$$\frac{1}{\lambda t}\sqrt{\frac{n}{2\pi}}\exp\left[-\frac{n}{2\lambda^2}\left(\frac{1}{t}\left[X-\frac{M_{tot}+m}{2\rho}\right]-m\tau\left[\frac{N-n}{2}-j\right]\right)^2\right]P_n(\tau) \quad (23)$$

Here

$$P_n(\tau)=\sqrt{2\pi n}\int du_1\cdots\int du_{n-1}\,\phi(u_1)\,\phi(u_2-u_1)\cdots\phi(u_{n-1}-u_{n-2})$$

$$\times\phi(-u_{n-1})\prod_{r=1}^{n-1}\theta\left[u_r+\frac{r(n-r)m\tau}{2\lambda}\right] \quad (24)$$

is the probability of merging of an isolated n-particle cluster into a single mass before time t, with

$$\phi(u)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right) \quad (25)$$

It is exactly this quantity which was the main object of our previous study.[4] In particular the important relation (6) has been derived therein.

In quite a similar way one can analyze the other two averages appearing in Eq. (20). We give the results here only for $t=t^*$ (or $\tau=0$), as our aim is to find the mass density at this characteristic moment. A straightforward calculation yields then the formulae

$$\left\langle\prod_{s=1}^{j}\theta[X-X_{j-s+1}^s(t^*)]\right\rangle=B_j\left(\frac{1}{\lambda t^*}\left[X-\frac{M_{tot}+m}{2\rho}\right]\right) \quad (26)$$

$$\left\langle\prod_{s=1}^{N-j-n}\theta[X_{n+1}^s(t^*)-X]\right\rangle=B_{N-n-j}\left(-\frac{1}{\lambda t^*}\left[X-\frac{M_{tot}+m}{2\rho}\right]\right) \quad (27)$$

where

$$B_j(u)=\int du_1\cdots\int du_j\,\phi(u_1)\,\phi(u_2-u_1)\cdots\phi(u_j-u_{j-1})\prod_{s=1}^{j}\theta(su+u_s)$$

$$\equiv E_W[u_s>-su,s=1,...,j] \quad (28)$$

As in ref. 4, u_s is interpreted as a Brownian path at discrete times $s=1,2,...,$ and the traditional notation $E_W[\cdots]$ for the Wiener measure has also been used here. Taking into account the relation (6) we eventually find that

the mass density at the gravitationally imposed finite time scale t^* in the continuum limit (7) equals

$$p(X; t^*) = M_{tot} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{j=0}^{N-n} \frac{1}{\lambda t^*} \sqrt{\frac{n}{2\pi}} \exp\left[-\frac{n}{2(\lambda t^*)^2} \left(X - \frac{M_{tot}}{2\rho} \right)^2 \right]$$

$$\times B_j\left(\frac{1}{\lambda t^*} \left[X - \frac{M_{tot}}{2\rho} \right] \right) B_{N-n-j}\left(-\frac{1}{\lambda t^*} \left[X - \frac{M_{tot}}{2\rho} \right] \right) \tag{29}$$

For symmetry reasons one can expect the macroscopic aggregation to take place at the central point of the originally uniform system. And indeed, it turns out that the mass density (29) becomes singular at this point in the continuum limit (7). In order to prove it we shall use now the Sparre Andersen theorem (Section XII.7 in ref. 6) which permits to determine the generating function[4]

$$p(z; u) = 1 + \sum_{n=1}^{\infty} z^n B_n(u) \tag{30}$$

One finds (see the discussion in Section 5 in ref. 4)

$$p(z; u) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} E_W[u_n > -nu]) \right) \tag{31}$$

It follows that

$$p(z; u) \, p(z; -u) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} (E_W[u_n > -nu] + E_W[u_n > nu]) \right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) = \frac{1}{1-z} \tag{32}$$

Moreover, it follows from (32) that (setting $B_0(u) = 1$)

$$\sum_{n=0}^{\infty} z^n B_n(u) \sum_{\ell=0}^{\infty} z^\ell B_\ell(u) = \sum_{N=0}^{\infty} z^N \sum_{\ell=0}^{N} B_\ell(u) B_{N-\ell}(-u) = \frac{1}{1-z}$$

which implies

$$\sum_{j=0}^{N-n} B_j(u) \, B_{N-n-j}(-u) = 1 \tag{33}$$

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[4] $B_n(u)$ here is the same as B_n in (42) of ref. 4 with the variable u playing the role of τ. Note that there is a minus sign missing in the equalities (46) and (48) of ref. 4.

Thus the Eq. (29) simplifies in a remarkable way to the form

$$\rho(X; t^*) = M_{tot} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\lambda t^*} \sqrt{\frac{n}{2\pi}} \exp\left[-\frac{n}{2(\lambda t^*)^2} \left(X - \frac{M_{tot}}{2\rho} \right)^2 \right] \quad (34)$$

The main conclusion from Eq. (34) is that at the moment t^* the mass density concentrates at the central point of the system $X = M_{tot}/2\rho$. Indeed, if $f(X)$ is a continuous function,

$$\int f(X)\, \rho(X; t^*)\, dX$$

$$= M_{tot} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int dY\, f\left(\frac{M_{tot}}{2\rho} + \frac{\lambda t^*}{\sqrt{n}}\, Y \right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Y^2}{2} \right)$$

$$= M_{tot}\, f\left(\frac{M_{tot}}{2\rho} \right)$$

leading to

$$\rho(X; t^*) = M_{tot}\, \delta\left(X - \frac{M_{tot}}{2\rho} \right) \quad (35)$$

In particular, the formula

$$\rho\left(\frac{M_{tot}}{2\rho}; t^* \right) \sim M_{tot} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\lambda t^*} \sqrt{\frac{n}{2\pi}}, \quad N \to \infty \quad (36)$$

shows that the mass density at $X = M_{tot}/2\rho$ (36) tends to infinity as $\sqrt{N}$.

Equation (35) is the main result of this Section. It tells us that even in a nonstatic case where the aggregating particles have initially a Gaussian velocity distribution the macroscopic mass is formed at the gravitational time scale $t = t^*$. From this point of view the situation does not differ from the static problem discussed in Section 1. However, the relation (6) clearly shows that the probability of complete aggregation at $t = t^*$ is still zero. This puts forward the question of the distribution of those masses which have not joint the central macroscopic body at $t = t^*$. The statistical distribution of this leftover dust, representing a nonextensive amount composed of microscopic fragments, is discussed in the next section.

3. STATISTICAL DISTRIBUTION OF MASSES

We first write down the probability distribution

$$\mu_k^N(X_1 V_1 n_1, ..., X_k V_k n_k; t)$$

for finding, at time t, an ordered configuration of k aggregates at positions X_i, $(X_1 < X_2 < \cdots < X_k)$ with velocities V_i and masses $M_{n_i} = mn_i$.

The aggregates originate from consecutive initial clusters made of $n_1, ..., n_k$ initial masses m and are found at the center of mass of these clusters

$$X_{n_i}(t) = \left(n_1 + n_2 + \cdots + n_{i-1} + \frac{n_i + 1}{2} \right) a + \frac{V_{n_i}}{n_i} t + \gamma m\, \Delta_{n_i} \frac{t^2}{2} \qquad (37)$$

with velocities

$$V_{n_i}(t) = \frac{V_{n_i}}{n_i} + \gamma m\, \Delta_{n_i} t \qquad (38)$$

The relations (37) and (38) generalize (11) and (12) to the splitting of the N equidistant initial particles into k consecutive clusters (here, $i = 1, ..., k$ does not index lattice sites, but the clusters themselves). In (37) and (38), V_{n_i} is the sum of the initial velocities of the particles belonging to the ith cluster and

$$m\Delta_{n_i} = m(n_{i+1} + \cdots + n_k - n_1 - \cdots - n_{i-1}) \qquad (39)$$

is the difference of the masses to the right and to the left of this cluster responsible for the force acting on it.

The distribution μ_k^N is obtained by averaging the set of kinematical constraints required for the realization of the desired event

$$\mu_k^N(X_1 V_1 n_1, ..., X_k V_k n_k; t)$$

$$= \left\langle \prod_{i=1}^{k-1} \theta(X_{i+1} - X_i) \prod_{i=1}^{k} [\delta(X_i - X_{n_i}(t))\, \delta(V_i - V_{n_i}(t))] \prod_{i=1}^{k} \Theta_{n_i}(t) \right\rangle$$

$$(40)$$

We always have $\sum_{i=1}^{k} n_i = N$, and the quantity $\Theta_{n_i}(t)$ represents the constraint (to be elaborated below) needed to ensure that the ith aggregate forms before time t. Both $X_{n_i}(t)$, $V_{n_i}(t)$ and $\Theta_{n_i}(t)$ are known expressions of the initial velocities and $\langle \cdots \rangle$ denotes as before the mean value with respect to the distribution (19), so μ_k^N can be calculated in principle. Notice that we did not include further constraints saying that the particles belonging to the adjacent clusters i and $i + 1$ do not perturb each other before t. This is not needed since the very fact that all initial particles in the $(i + 1)$th

cluster are found right of those in the ith cluster at time t already implies that no particles of the two groups have collisioned before t (otherwise a double crossing between trajectories would have occured, which is not possible in our dynamics).

To simplify the discussion, we shall merely be interested in the mass distribution by integrating out positions and velocities

$$\mu_k^N(n_1, n_2,..., n_k; t)$$

$$= \int dX_1 \cdots dX_k \int dV_1 \cdots dV_k \, \mu_k^N(X_1 V_1 n_1,..., X_k V_k n_k; t)$$

$$= \left\langle \prod_{i=1}^{k-1} \theta(X_{n_{i+1}}(t) - X_{n_i}(t)) \prod_{i=1}^{k} \Theta_{n_i}(t) \right\rangle \tag{41}$$

The conditions for forming the ith aggregate before t are the same as (13), i.e., $X_{n_i}^r(t) > X_{n_i}^{n_i-r}(t)$, $r = 1, 2,..., n_i - 1$, where $X_{n_i}^r(t)$ is the position of the subcluster of the first r particles in the ith cluster

$$X_{n_i}^r(t) = \left(n_1 + \cdots + n_{i-1} + \frac{r+1}{2}\right) a + \frac{1}{r} \sum_{s=1}^{r} v_s t$$

$$+ \gamma m (n_i - r + n_{i+1} + \cdots + n_k - n_1 - \cdots - n_{i-1}) \frac{t^2}{2} \tag{42}$$

Here the velocities of the initial particles in this cluster have been simply labelled $v_1, v_2,..., v_{n_i}$ and $V_{n_i} = \sum_{s=1}^{n_i} v_s$. Hence, as in (21), the constraint is

$$\Theta_{n_i}(t) = \prod_{i=1}^{n_i-1} \theta[X_{n_i}^r(t) - X_{n_i}^{n_i-r}(t)]$$

$$= \prod_{i=1}^{n_i-1} \theta\left(\sum_{s=1}^{r} v_s - \frac{r}{n_i} V_{n_i} + \frac{r(n_i-r)}{2} m\tau\right) \tag{43}$$

We see from (37) that the factor

$$\prod_{i=1}^{k-1} \theta(X_{n_{i+1}}(t) - X_{n_i}(t)) = \prod_{i=1}^{k-1} \theta\left(\frac{V_{n_{i+1}}}{n_{i+1}} - \frac{V_{n_i}}{n_i} - \frac{n_i + n_{i+1}}{2} m\tau\right) \tag{44}$$

in (41) depends only on the initial center of mass velocities V_{n_i}/n_i of the clusters. Since the initial velocity distribution factorizes, we can perform the

integration independently for each cluster, except for the variables V_{n_i}/n_i that are coupled through (44). Introducing for each cluster the change of variables (22) one obtains as in (21)–(24) that the ith cluster contributes to the total integration on velocities in (41) as

$$\int dv_1 \cdots dv_{n_i} \, \phi_\lambda(v_1) \cdots \phi_\lambda(v_{n_i}) \prod_{i=1}^{n_i-1} \theta \left(\sum_{s=1}^{r} v_s - \frac{r}{n_i} V_{n_i} + \frac{r(n_i-r)}{2} m\tau \right) \cdots$$

$$= P_{n_i}(\tau) \frac{1}{\sqrt{2\pi n_i} \, \lambda} \int dV_{n_i} \exp \left(-\frac{V_{n_i}}{2 n_i \lambda^2} \right) \cdots \qquad (45)$$

Taking (44) and (45) into account in (41) and setting $U_i = V_{n_i}/\lambda \sqrt{n_i}$ leads to the final result

$$\mu_k^N(n_1, n_2, \dots, n_k; t) = \left(\prod_{i=1}^{k} P_{n_i}(\tau) \right) \int dU_1 \cdots dU_k \, \phi(U_1) \cdots \phi(U_k)$$

$$\times \prod_{i=1}^{k-1} \theta \left(\frac{U_{i+1}}{\sqrt{n_{i+1}}} - \frac{U_i}{\sqrt{n_i}} - \frac{n_i + n_{i+1}}{2\lambda} m\tau \right) \qquad (46)$$

The first factor is the probability of formation of independent aggregates, whereas the second factor represents the correlations introduced between them by the gravitational forces.

We now draw some important conclusions from the formula (46). We say that the ith aggregate is macroscopic if the number of its constituents $n_i = \eta N$, $0 < \eta \leq 1$, is a non vanishing fraction η of the total number of initial particles as $N \to \infty$; its mass is then $M_i = m n_i = \eta M_{\text{tot}}$.

(i) Macroscopic Mass

According to the form of the mass density (35) at $t = t^*$, it is clear that there can be only one macroscopic aggregate for $t > t^*$. Let us recover this result using (46). The arguments of the θ-functions in (46) are denoted by

$$w_{i,i+1} = \frac{U_{i+1}}{\sqrt{n_{i+1}}} - \frac{U_i}{\sqrt{n_i}} - \frac{n_i + n_{i+1}}{2\lambda} m\tau \qquad (47)$$

The integral in (46) is carried out on the domain $\mathscr{D}$ of the variables U_i defined by $w_{i,i+1} \geq 0$. Suppose that the masses of the two clusters j and $j + \ell$ become macroscopic, i.e., $n_j \to \infty, n_{j+\ell} \to \infty$ with $m n_j = M_j > 0$,

$mn_{j+\ell} = M_{j+\ell} > 0$ and n_i remains finite for $i \neq j, j+\ell$. Then as $N \to \infty$,

$$w_{j,j+1} = \frac{U_{j+1}}{\sqrt{n_{j+1}}} - \frac{M_j}{2\lambda} \tau \tag{48}$$

$$w_{j+\ell-1,j+\ell} = -\frac{U_{j+\ell-1}}{\sqrt{n_{j+\ell-1}}} - \frac{M_{j+\ell}}{2\lambda} \tau \tag{49}$$

The relation (48) together with the conditions $w_{i,i+1} \geq 0$ for $\tau > 0$ imply $U_{j+1} > 0,..., U_{j+\ell-1} > 0$, but (49) implies also $U_{j+\ell-1} < 0$; thus the integration domain $\mathscr{D}$ shrinks to zero as $N \to \infty$ and the corresponding probability vanishes. The argument is the same when more than two masses become macroscopic.

We calculate now the probability to have one macroscopic mass, say $n_j = N - \sum_{i \neq j}^{k} n_i$, holding the other aggregates n_i, $i \neq j$ finite, defined by

$$\mu_k(n_1,..., n_{j-1}, M_{\text{tot}}, n_{j+1},..., n_k; t)$$

$$= \lim_{N \to \infty} \mu_k^N \left(n_1,..., n_{j-1}, N - \sum_{i \neq j}^{k} n_i, n_{j+1},..., n_k; t \right) \tag{50}$$

Notice that in the macroscopic limit, the mass $M_{n_j} = M_{\text{tot}} - m \sum_{i \neq j}^{k} n_i$ becomes infinitesimally close to the total mass: $M_{n_j} \to M_{\text{tot}}$. We make two observations on the probability $P_n(\tau)$, $\tau > 0$. If n is fixed

$$\lim_{N \to \infty} P_n(\tau) = P_n(0) = \frac{1}{n} \tag{51}$$

since this amounts to let $m = M_{\text{tot}}/N \to 0$ in (24). If $n = N - q$ with q a fixed integer

$$\lim_{N \to \infty} P_{N-q}(\tau) = P(\tau) \tag{52}$$

where

$$P(\tau) = \exp(-A(\tau)),$$

$$\text{with} \quad A(\tau) = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{(M_{\text{tot}}/2\lambda)\tau}^{\infty} \phi(\sqrt{n}\, y)\, dy \tag{53}$$

is the probability of merging of the total number of particles into the single mass M_{tot}. This is precisely the function determined in the proposition found in Section 5 of ref. 4, because again $m(N - q) = M_{\text{tot}} - mq \to M_{\text{tot}}$ as $N \to \infty$.[5]

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[5] In this proposition the quantity $M_{\text{tot}}/2\lambda$ was set equal to one.

Finally, as $N \to \infty$ the arguments $w_{i,i+1}$ (47) tend to

$$w_{i,i+1} = \frac{U_{i+1}}{\sqrt{n_{i+1}}} - \frac{U_i}{\sqrt{n_i}}, \qquad i \neq j, \qquad i+1 \neq j$$

$$w_{j,j+1} = \frac{U_{j+1}}{\sqrt{n_{j+1}}} - \frac{M_{tot}}{2\lambda}\tau \tag{54}$$

$$w_{j,j-1} = \frac{U_{j-1}}{\sqrt{n_{j-1}}} - \frac{M_{tot}}{2\lambda}\tau$$

When (51), (52) and (54) are taken into account in (46) (changing also U_i into $-U_i$), one finds that the limit (50) is

$$\mu_k(n_1,...,n_{j-1}, M_{tot}, n_{j+1},..., n_k; t)$$

$$= Q_{j-1}(n_1,...,n_{j-1};\tau) \, P(\tau) \, Q_{k-j}(n_k,...,n_{j+1};\tau) \tag{55}$$

with

$$Q_{j-1}(n_1,...,n_j;\tau)$$

$$= \frac{1}{n_1 \cdots n_j} \int dU_1 \cdots dU_j \, \phi(U_1) \cdots \phi(U_j)$$

$$\times \theta\left(\frac{U_1}{\sqrt{n_1}} - \frac{U_2}{\sqrt{n_2}}\right) \cdots \theta\left(\frac{U_{j-1}}{\sqrt{n_{j-1}}} - \frac{U_j}{\sqrt{n_j}}\right) \theta\left(\frac{U_j}{\sqrt{n_j}} - \frac{M_{tot}}{2\lambda}\tau\right) \tag{56}$$

The interpretation of (56) is clear: the factor $Q_{j-1}(Q_{k-j})$ is the probability to find, left (right) of the macroscopic mass, j $(k-j)$ aggregates made of a finite number of initial masses, that we call now fragments. We shall show in paragraph (ii) below that the probabilities (55) sum up to one. Therefore the only configurations that can occur after t^* consist of a single macroscopic mass together with a dust of such fragments. One should notice that non macroscopic pieces of matter of the order mN^ν, $0 < \nu < 1$, do not appear after t^*.

(ii) Statistics of Fragments

The probability to have a configuration of exactly k bodies after t^* (i.e., one macroscopic mass and $k-1$ fragments) is

$$\mu_k(t) = \sum_{j=1}^{k} \sum_{n_1,...,n_{j-1},n_{j+1},...,n_k = 1}^{\infty} \mu_k(n_1,...,n_{j-1}, M_{tot}, n_{j+1},..., n_k; t) \tag{57}$$

After the change of variables $y_i = ((U_i/\sqrt{n_i}) - (M_{tot}/2\lambda)\,\tau)$, one finds from (56)

$$\sum_{n_1,\ldots,n_j}^{\infty} Q_{j-1}(n_1,\ldots,n_j;\tau)$$

$$= \int_{y_1 \geqslant \cdots y_{j-1} \geqslant y_j \geqslant 0}^{\infty} dy_1 \cdots dy_j \prod_{i=1}^{j} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \phi\left(\sqrt{n}\left[y_i + \frac{M_{tot}}{2\lambda}\tau\right]\right)$$

$$= \frac{1}{j!}\left(\frac{A(\tau)}{2}\right)^j \tag{58}$$

where $A(\tau)$ is the function defined in (53). Hence from (55), (57), (58) and (53) one obtains the result

$$\mu_k(t) = P(\tau)\left(\frac{A(\tau)}{2}\right)^{k-1} \sum_{j=1}^{k} \frac{1}{(j-1)!\,(k-j)!}$$

$$= \frac{(A(\tau))^{k-1}}{(k-1)!}\exp(-A(\tau)), \qquad t \geqslant t^* \tag{59}$$

As claimed above, the $\mu_k(t)$ satisfy the normalization relation

$$\sum_{k=1}^{\infty} \mu_k(t) = 1$$

The distribution of the number $k-1$ of fragments is Poissonian for all times $t > t^*$. We find therefore that $A(\tau)$ appearing in (53) has the interpretation of the mean number of fragments. $A(\tau)$ tends to zero as a Gaussian when $\tau \to \infty$ and diverges as $-2\ln((M_{tot}/2\lambda)\,\tau)$ when $\tau \to 0$ (see Eq. (52) in ref. 4).

As a particular exemple, we write down from (55) and (56) the probability of survival of a fragment of size n

$$\mu_2(M_{tot}, n; t) = \frac{1}{2}P(\tau)\left(1 - \text{erf}\left(\frac{\sqrt{n}\,M_{tot}}{\sqrt{8}}\tau\right)\right)$$

$$\sim \begin{cases} \dfrac{1}{2}\left(\dfrac{M_{tot}}{2\lambda}\tau\right)^2 & \text{if } \tau \to 0 \\[4mm] \dfrac{\sqrt{8}}{\sqrt{\pi n}\,M_{tot}\tau}\exp\left(-\dfrac{n(M_{tot}\tau)^2}{8}\right) & \text{if } \tau \to \infty \end{cases} \tag{60}$$

where erf is the error function.

The general position and velocity dependent distributions (40) for $t > t^*$ can also be written down more explicitly in the macroscopic limit. In particular, one has $\mu_1(X_1 V_1 M_{tot}; t) = \delta(X_1 - (M_{tot}/2\rho)) \, \delta(V_1) \, P(\tau)$. The macroscopic mass is found at rest at the position $M_{tot}/2\rho$ without fluctuations, as expected. More generally, there are Gaussian small probabilities to find fragments far away from this point.

So far we have given a full description of the state after t^*: here the structure is simple since the weight of typical mass configurations is given by the set of distributions (45) with k finite. When $t < t^*$ the situation is more complex. Indeed, we have

$$\lim_{N \to \infty} \mu_k^N(n_1, n_2, ..., n_k; t) = 0, \qquad k = 1, 2, ..., \quad t < t^* \tag{61}$$

since necessarily at least one of the n_i tends to infinity and we know then from (6) that for $\tau \leqslant 0$, $P_{n_i}(\tau) \leqslant 1/n \to 0$ (all the other factors in (46) are bounded by 1). Hence there remain always infinitely many aggregates as $N \to \infty$, and the weight of typical configurations will be given by the distributions $\mu_k^N(n_1, ..., n_k; t)$, $k \to \infty$, involving infinitely many bodies. Computer simulations indicate that after a short transient time, typical configurations consist of approximately $\sqrt{N}$ aggregates, each of them having a mass of the order $m\sqrt{N}$. Thus we may conjecture that the distributions in this range, i.e., $\sum_{n_1, ..., n_k = c_1\sqrt{N}}^{c_2\sqrt{N}} \mu_k^N(n_1, ..., n_k; t)$ with $k \sim \sqrt{N}$, should have a non vanishing limit as $N \to \infty$.

As far as the density (20) is concerned, we anticipate that it converges for $t < t^*$ to an absolutely continuous function, namely the uniform density (8) as in the static model. This would be consistent with the numerical observation that after a short time during which most of the initial kinetic energy is dissipated by inelastic collisions, the subsequent evolution is dominated by the gravitational forces. The result of simulations is discussed in the next Section. Analytic proofs of these conjectures would complete the study of the dynamical phase transition that occurs at t^* between a spatially extended and an aggregated phase of matter.

4. NUMERICAL SIMULATIONS

In this section we present the results of numerical simulations performed to determine the rate of formation of macroscopic masses for a system with a *finite* number N of particles evolving according to the model described above. In particular, we analyze the scaling to the continuum limit of the probability $P_N^\eta(t)$ for the formation of a macroscopic mass $\eta N m$, $0 < \eta \leqslant 1$, before time t. We also study the time evolution of the kinetic energy.

Numerical simulations on this model are particularly simple, compared with their counterparts in higher dimensions, because the equations of motion can be analytically integrated between successive collisions. The simulation then reduces to keeping track of the particle masses, coordinates and velocities created in successive collisions.

In going to the continuum limit while keeping the density constant, different scalings of the initial conditions are possible. Here, for reasons of numerical accuracy, we increase the number of particles N, while keeping the distance between them constant: $a = 1$, and also putting $m = 1$, so that $\rho = M/L = nm/na = 1$. Further, we choose initial velocities with a Gaussian distribution of variance $\lambda = N/2$. This procedure is equivalent to the continuum limit considered in the preceding sections (see (7) and also ref. 4).

In Fig. 1 we show the mass formation probability $P^\eta_N(t)$ for $N = 1000$ as a function of time, averaged over 10000 initial configurations, with $\eta = 0.1, 0.2,..., 0.9, 0.99, 1.0$ from left to right.

Note that $P^\eta_N(t)$ clusters around the Heaviside function for $\eta < 1$, while the probability of total mass aggregation $\eta = 1$ follows the separate limiting curve given by (53).

In order to study the scaling of $P^\eta_N(t)$ curves as a function of the particle number N, we obtained the results for the initial values $N = 10 \cdot 2^r$, $r = 0, 1,..., 9$, averaged over 1000 initial configurations. Fixing $\eta = 0.5$, we display in Fig. 2 the probability curves for increasing numbers of initial particles. They tend to the Heaviside function as $N \to \infty$.

Having in view a quantitative study of this scaling we consider the time deviation $|t^\eta_\beta(N) - t^*|$ for increasing N, where $t^\eta_\beta(N)$ is the time at which the probability P^η_N acquires the value β, i.e., $P^\eta_N(t^\eta_\beta(N)) = \beta$, $\beta = 0.25, 0.5, 0.75$, $\eta = 0.5$.
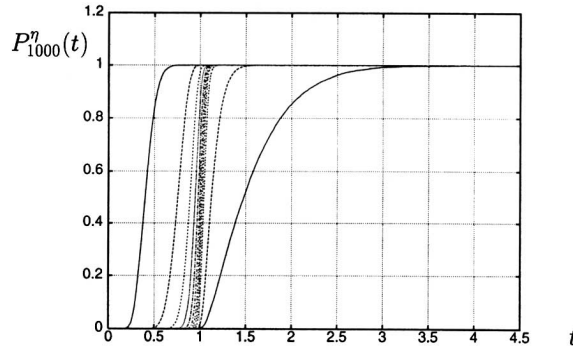

Fig. 1.   Mass formation probability $P^\eta_N(t)$, $N = 1000$, $\eta = 0.1, 0.2,..., 0.9, 0.99, 1.0$ from left to right.
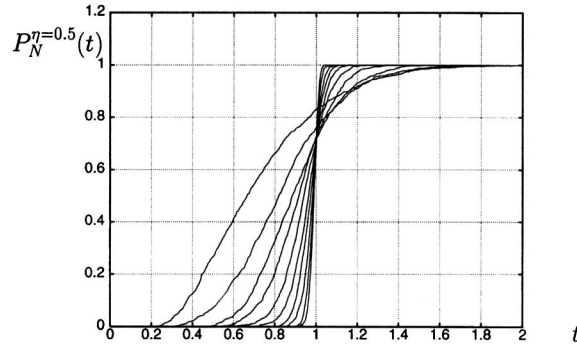
Fig. 2. Mass formation probability $P_N^{0.5}(t)$, $N = 10 \cdot 2^r$, $r = 0, 1,..., 9$, from left to right.

Figure 3 reveals the power law

$$|t_\beta^\eta(N) - t^*| \simeq \frac{1}{\sqrt{N}}$$

The same behavior holds also for other values of η.

In order to get a deeper understanding of the dynamics of aggregation we analyze the evolution of the kinetic energy $E_{\mathrm{kin}}^N(t)$. If we started from a static initial configuration, we would simply find a parabolic law

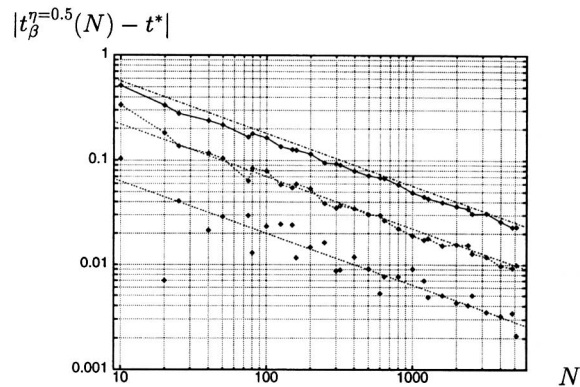$$E_{\mathrm{kin}}^{N,\,\mathrm{stat}}(t) = \frac{\gamma^2}{6} M_{\mathrm{tot}}^3 t^2$$

Fig. 3. Log-log plot of $|t_\beta^\eta(N) - t^*|$, for $\beta = 0.25, 0.5, 0.75$ from up to down.
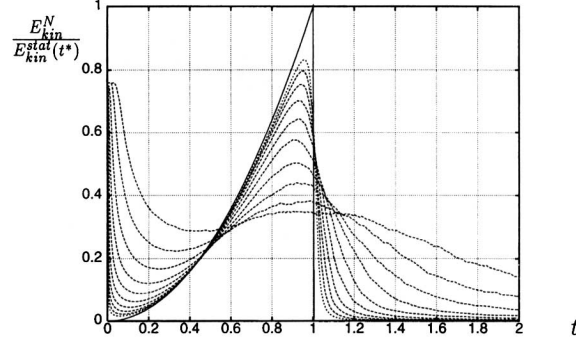
Fig. 4. Normalized kinetic energy as a function of time for $N = 10 \cdot 2^r$, $r = 0, 1,..., 9$. The parabolic limit curve represents the normalized kinetic energy of the continuous static initial configuration.

In Fig. 4 the ratio $E_{kin}^N(t)/E_{kin}^{N,\,stat}(t^*)$ has been plotted. This ratio approaches the normalized parabola as $N \to \infty$. One can look at the local minimum of the curves as corresponding to the time scale of almost total dissipation of the kinetic energy due to initially numerous inelastic collisions. For $N \to \infty$, the location of the minimum approaches the initial moment $t = 0$ according to the power law $N^{-1/4}$. Hence in the continuum limit the system gets instantaneously cooled down and the subsequent evolution is dominated by gravity.

We also observe that at the time when the kinetic energy attains its minimum:

(i)   the average size of the formed masses scales as $\sim m \sqrt{N}$

(ii)   the velocity distribution remains Gaussian with an effective standard deviation $\lambda_{eff} \sim \lambda N^{-1/4}$.

It strongly suggests that before t^* the density of mass in the continuum limit should also coincide with that of the static model (8).

Finally, in order to study the sensitivity of our results to variations of the initial distributions, we have led the same simulations for interparticle distances following a Poissonian distribution and a Gaussian initial velocity distribution. We confirmed that all the results presented above remained identical, indicating a certain generality of the initial kinetic energy dissipation process.

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REFERENCES

1. Ya. B. Zeldovich, *Astron. Astrophys.* **5**:84 (1970).
2. M. Vergassola, B. Dubrulle, U. Frisch, and A. Noullez, *Astron. Astrophys.* **289**:325 (1994).
3. E. Weinan, Yu. G. Rynov, and Ya. G. Sinai, *Commun. Math. Phys.* **177**:349 (1996).
4. Ph. A. Martin and J. Piasecki, *J. Stat. Phys.* **84**:837 (1996).
5. Ph. A. Martin and J. Piasecki, *J. Stat. Phys.* **76**:447 (1994).
6. W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1968).